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Stability of the second order partial differential equations

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Abstract

We say that a functional equation (ξ) is stable if any function g satisfying the functional equation (ξ) approximately is near to a true solution of (ξ).

In this paper, by using Banach's contraction principle, we prove the stability of nonlinear partial differential equations of the following forms:

$$\begin{cases} y_x(x,t) = f(x,t,y(x,t)), \\ ay_x(x,t) + by_t(x,t) = f(x,t,y(x,t)), \\ p(x,t)y_{xt}(x,t) + q(x,t)y_t(x,t) + p_t(x,t)y_x(x,t) - p_x(x,t)y_t(x,t) = f(x,t,y(x,t)), \\ p(x,t)y_{xx}(x,t) + q(x,t)y_x(x,t) = f(x,t,y(x,t)). \end{cases}$$

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1. Introduction

Let X be a normed space over a scalar field $\mathbb{K}$, and let I be an open interval. Assume that, for any function $f : I \to X$ ($y = f(x)$) satisfying the differential inequality

$$||a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) + h(t)|| \le \varepsilon$$

for all $t \in I$, where $\varepsilon \ge 0$, there exists a function $f_0 : I \to X$ satisfying

$$\begin{cases} y_0 = f_0(x), \\ a_n(t)y_0^{(n)}(t) + a_{n-1}(t)y_0^{(n-1)}(t) + \cdots + a_1(t)y'_0(t) + a_0(t)y_0(t) + h(t) = 0 \end{cases}$$

and $||f(t) - f_0(t)|| \le K(\varepsilon)$ for any $t \in I$.

Then we say that the above differential equation has the *Hyers-Ulam stability*. If the above statement is also true, then we replace ε and $K(\varepsilon)$ by $\phi(t)$ and $\varphi(t)$, where $\phi, \varphi : I \to [0, \infty)$ are functions not depending on f and f_0 explicitly, then we say that the corresponding differential equation has the *Hyers-Ulam-Rassias stability* or the *generalized Hyers-Ulam stability*.

In 1998, the Hyers-Ulam stability of differential equation $y' = y$ was first investigated by Alsina and Ger [1]. In 2002, this result has been generalized by Takahasi et al. [2] for the Banach space-valued differential equation $y' = \lambda y$. In 2005, Jung [3] proved the generalized Hyers-Ulam stability of a linear differential equation of the first order. For more results on stability of differential equations, see also [4-7] and [8] and, for more details on the Hyers-Ulam stability and related topics, the readers refer to [9-17] and [18-20].

In this paper, we prove the Hyers-Ulam-Rassias stability of the following partial differential equations:

(1) The first order nonlinear partial differential equation:

$$y_x(x,t) = f(x,t,y(x,t));$$

(2) The first order nonlinear partial differential equation:

$$ay_x(x,t) + by_t(x,t) = f(x,t,y(x,t))$$

for all $a, b \in \mathbb{R}$;

(3) The second order nonlinear partial differential equation:

$$p(x,t)y_{xx}(x,t) + q(x,t)y_x(x,t) = f(x,t,y(x,t)) \tag{1.1}$$

under the following condition:

$$p_{xx}(x,t) = q_x(x,t). \tag{1.2}$$

The differential equation (1.1) is the second order nonlinear partial differential equation, and we call it *exact* if the condition (1.2) holds.

(4) The mixed type second order nonlinear partial differential equation:

$$p(x,t)y_{xt}(x,t) + q(x,t)y_t(x,t) + p_t(x,t)y_x(x,t) - p_x(x,t)y_t(x,t) = f(x,t,y(x,t))$$

under the following condition:

$$p_{xt}(x,t) = q_t(x,t).$$

Theorem 1.1. (Banach's Contraction Principle) *Let* (X, d) *be a complete matric space and* $T : X \to X$ *be a contraction, that is, there exists* $\alpha \in [0,1)$ *such that*

$$\mathrm{d}(Tx, Ty) \le \alpha \mathrm{d}(x, y)$$

for all $x, y \in X$. *Then, there exists a unique* $a \in X$ *such that* $Ta = a$. *Moreover,* $a = \lim_{n\to\infty} T^n x$ *and*

$$\mathrm{d}(a, x) \le \frac{1}{1-\alpha}\mathrm{d}(x, Tx)$$

for all $x \in X$.

2. Main results

In this section, let $I = [a, b]$ be a closed interval with $a < b$ and $C(I \times I) = \{f : I \times I \to \mathbb{R} : f \text{ is continuous}\}$. For the sake of convenience, assume that all the integrals and all the derivatives exist.

Theorem 2.1. *Let $c \in I$ $\phi : I \times I \to (0, \infty)$ be a continuous function, $L : I \times I \to [1,\infty)$ be an integrable function and $K : I \times I \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exists $0 < \beta < 1$ such that*

$$\int_c^x L(\tau, t)\varphi(\tau, t)\mathrm{d}\tau < \beta\varphi(x, t); \tag{2.1}$$

$$|K(x, t, u(x, t)) - K(x, t, v(x, t))| \le L(x, t)|u(x, t) - v(x, t)| \tag{2.2}$$

for all $x, t \in I$ and $u, v \in C(I \times I)$. Let $y : I \times I \to \mathbb{R}$ be such that

$$|y_x(x, t) - K(x, t, y(x, t))| \le \varphi(x, t) \tag{2.3}$$

for all $x, t ⌊ I$. Then, there exists a unique continuously differentiable function $y_0 : I \times I \to \mathbb{R}$ such that

$$y_0(x, t) = y(c, t) + \int_c^x K(\tau, t, y_0(\tau, t))\mathrm{d}\tau$$

(consequently, y_0 is a solution to $y_x(x, t) = K(x, t, y(x,t))$) and

$$|y(x, t) - y_0(x, t)| \le \frac{\beta}{1-\beta}\varphi(x, t)$$

for all $x, t ⌊ I$.

Proof. Let X be the set of all continuously differentiable functions $u : I \times I \to \mathbb{R}$. We define a metric d and an operator T on X as follows, respectively:

$$\mathrm{d}(u, v) = \sup_{x,t\in I}\frac{|u(x, t) - v(x, t)|}{\varphi(x, t)}$$

and the operator

$$(Tu)\ (x, t) = y(c, t) + \int_c^x K(\tau, t, u(\tau, t))\mathrm{d}\tau$$

for all $u \in X$. Using (2.1) and (2.2), we have

$$\begin{aligned}
\mathrm{d}(Tu, Tv) &= \sup_{x,t\in I}\frac{\left|\int_c^x \left[K(\tau, t, u(\tau, t)) - K(\tau, t, v(\tau, t))\right]\mathrm{d}\tau\right|}{\varphi(x, t)}\\
&\le \sup_{x,t\in I}\frac{\int_c^x L(\tau, t)|u(\tau, t) - v(\tau, t)|\mathrm{d}\tau}{\varphi(x, t)}\\
&= \sup_{x,t\in I}\left[\frac{\int_c^x L(\tau, t)\varphi(\tau, t)\frac{|u(\tau,t)-v(\tau,t)|}{\varphi(\tau,t)}\mathrm{d}\tau}{\varphi(x, t)}\right]\\
&\le \sup_{x,t\in I}\frac{\int_c^x L(\tau, t)\varphi(\tau, t)\ \sup_{\tau,t\in I}\frac{|u(\tau,t)-v(\tau,t)|}{\varphi(\tau,t)}\mathrm{d}\tau}{\varphi(x, t)}\\
&= \mathrm{d}(u, v)\ \sup_{x,t\in I}\frac{\int_c^x L(\tau, t)\varphi(\tau, t)\mathrm{d}\tau}{\varphi(\tau, t)}\\
&\le \beta\mathrm{d}(u, v).
\end{aligned}$$

Now, by Theorem 1.1, there exists a unique $y_0 \in X$ such that $Ty_0 = y_0$, that is,

$$y_0(x, t) = y(c, t) + \int_c^x K(\tau, t, y_0(\tau, t))\mathrm{d}\tau.$$

Moreover, by Theorem 1.1, we have

$$\mathrm{d}(y_0, y) \leq \frac{1}{1-\beta}\mathrm{d}(y, Ty) \tag{2.4}$$

for all $y \in X$. It follows from (2.3) that

$$-\varphi(x, t) \leq y_x(x, t) - K(x, t, y(x, t)) \leq \varphi(x, t)$$

for all $x, t \in I$. If we integrate each term in the above inequality from c to x, then we get

$$\begin{aligned} \left| y(x, t) - \left(y(c, t) - \int_c^x K(\tau, t, y(\tau, t) \right) \mathrm{d}\tau \right| &\leq \int_c^x \varphi(\tau, t)\mathrm{d}\tau \\ &\leq \int_c^x L(\tau, t)\varphi(\tau, t)\mathrm{d}\tau \\ &\leq \beta\varphi(x, t). \end{aligned}$$

Now, we have

$$\frac{|y(x, t) - (Ty)\ (x, t)|}{\varphi(x, t)} \leq \beta \Rightarrow \sup_{x,t\in I}\frac{|y(x, t) - (Ty)\ (x, t)|}{\varphi(x, t)} \leq \beta.$$

Thus, we get

$$\mathrm{d}(y, Ty) \leq \beta. \tag{2.5}$$

Therefore, by (2.4) and (2.5), we see that

$$|y(x, t) - y_0(x, t)| \leq \frac{\beta}{1-\beta}\varphi(x, t)$$

for all $x, t \in I$. This completes the proof. □

Theorem 2.2. *Let $c \in I$, p, $q : I \times I \to \mathbb{R}$ be continuous functions with $p(x, t) \neq 0$ for all $x, t \in I$, $\phi : I \times I \to (0, \infty)$ be a continuous function, $L : I \times I \to [1, \infty)$ be an integrable function, and $f : I \times I \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exists $0 < \beta < 1$ such that*

$$\int_c^x L(\tau, t)\varphi(\tau, t)\mathrm{d}\tau \leq \beta\varphi(x, t);$$

$$h(c, t) = -[p(c, t)y_x(c, t) - p_x(c, t)y(c, t) + q(c, t)y(c, t)];$$

$$\begin{aligned} &K(x, t, y(x, t)) \\ &= -(p(x, t))^{-1}\left[(p_x(x, t) - q(x, t))y(x, t) + h(c, t) - \int_c^x f(\tau, t, y(\tau, t)\mathrm{d}\tau\right] \end{aligned}$$

and

$$|K(x,t,u(x,t)) - K(x,t,v(x,t))| \le L(x,t)|u(x,t) - v(x,t)|$$

for all c, x, $t \in I$ *and* h, u, v, $y \in C(I \times I)$. *Let* $y : I \times I \to \mathbb{R}$ *be a function such that:*

$$|p(x,t)y_{xx}(x,t) + q(x,t)y_x(x,t) - f(x,t,y(x,t))| \le \varphi(x,t) \tag{2.7}$$

for all x, $t \in I$ *and* (1.2) *holds. Then, there exists a unique solution* $y_0 : I \times I \to \mathbb{R}$ *of* (1.1) *such that*

$$|y(x,t) - y_0(x,t)| \le \frac{\beta}{1-\beta}\varphi(x,t).$$

Proof. It follows from (1.2) and (2.7) that

$$\begin{aligned} & |p(x,t)y_{xx}(x,t) + q(x,t)y_x(x,t) - f(x,t,y(x,t))| \\ = & \ |(p(x,t)y_x(x,t) - p_x(x,t)y(x,t))_x + (q(x,t)y(x,t))_x \\ & + [p_{xx}(x,t) - q_x(x,t)]y(x,t) - f(x,t,y(x,t))| \\ = & \ |(p(x,t)y_{xx}(x,t) - p_x(x,t)y(x,t))_x + (q(x,t)y(x,t))_x - f(x,t,y(x,t))| \\ \le & \ \varphi(x,t). \end{aligned}$$

Thus, we have

$$\begin{aligned} & -\varphi(x,t) \\ \le & \left(p(x,t)y_{xx}(x,t) - p_x(x,t)y(x,t)\right)_x + (q(x,t)y(x,t))_x - f(x,t,y(x,t)) \\ \le & \ \varphi(x,t). \end{aligned} \tag{2.8}$$

By using (2.8), we get

$$\begin{aligned} & |p(x,t)y_x(x,t) - p_x(x,t)y(x,t) + q(x,t)y(x,t) + h(c,t) - \int_c^x f(\tau,t,y(\tau,t))\mathrm{d}\tau| \\ = & \ |p(x,t)| \left| y_x(x,t) + (p(x,t))^{-1}\left((q(x,t) - p_x(x,t))y(x,t) + h(c,t) \right.\right. \\ & \left.\left. - \int_c^x f(\tau,t,y(\tau,t))\mathrm{d}\tau\right)\right| \\ \le & \int_c^x \varphi(\tau,t)\mathrm{d}\tau, \end{aligned} \tag{2.9}$$

where

$$h(c,t) = -[p(c,t)y_x(c,t) - p_x(c,t)y(c,t) + q(c,t)y(c,t)].$$

From (2.9), it follows that

$$\begin{aligned} & \left| y_x(x,t) + (p(x,t))^{-1}\left((q(x,t) - p_x(x,t))y(x,t) + h(c,t) - \int_c^x f(\tau,t,y(\tau,t))\mathrm{d}\tau\right)\right| \\ \le & \ |p(x,t)|^{-1}\int_c^x \varphi(\tau,t)\mathrm{d}\tau. \end{aligned}$$

From $p(x,t) = \frac{p(x,t)\,[1+(p(x,t))^2]}{1+(p(x,t))^2}$, without less of generality, we can assume that $|p(x, t)| \geq 1$.

Now, By putting

$$\begin{aligned} & K(x,t,y(x,t)) \\ &= -(p(x,t))^{-1}\left[(p_x(x,t) - q(x,t))y(x,t) + h(c,t) - \int_c^x f(\tau,t,y(\tau,t)\mathrm{d}\tau\right] \end{aligned}$$

in the above inequality, we get

$$\begin{aligned} |y_x(x,t) - K(x,t,y(x,t))| &\leq |p(x,t)|^{-1}\int_c^x \varphi(\tau,t)\mathrm{d}\tau \\ &\leq \int_c^x \varphi(\tau,t)\mathrm{d}\tau \\ &\leq \int_c^x L(\tau,t)\varphi(\tau,t)\mathrm{d}\tau \\ &\leq \beta\varphi(x,t) \\ &\leq \varphi(x,t). \end{aligned}$$

Thus, the conclusions of the Theorem follows from Theorem 2.1. This completes the proof. □

If (1.1) is multiplied by a function $\mu(x, t)$ such that the resulting equation is exact, that is,

$$\mu(x,t)[p(x,t)y_{xx}(x,t) + q(x,t)y_x - f(x,t,y(x,t))] = 0 \tag{2.10}$$

and

$$(\mu(x,t)p(x,t))_{xx} - (q(x,t)\mu(x,t))_x = 0, \tag{2.11}$$

then we say that $\mu(x, t)$ is an *integrating factor* of the partial differential equation (1.1).

Corollary 2.3. *Let $p, q, \mu : I \times I \to \mathbb{R}$ be continuous functions such that $p(x, t) \neq 0$ and $\mu(x, t) \neq 0$ for all $x, t \in I$, and (2.10) holds. Assume that $c \in I$, $L : I \times I \to [1, \infty)$ is an integrable function and $f : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Suppose that there exists $0 < \beta < 1$ such that*

$$\int_c^x L(x,t)\varphi(\tau,t)\mathrm{d}\tau \leq \beta\varphi(x,t);$$

$$h(c,t) = -\left[\mu(c,t)p(x,t)y_x(c,t) - (\mu p)_x(c,t)y(c,t) + \mu(c,t)q(c,t)y(c,t)\right];$$

$$\begin{aligned} K(x,t,y(x,t)) = &-(\mu(x,t)p(x,t))^{-1}\Big[(\mu(x,t)q(x,t) - (\mu q)_x(x,t))y(x,t) \\ &+ \left(h(c,t) - \int_c^x \mu(\tau,t)f(\tau,t,y(\tau,t))\mathrm{d}\tau\right)\Big] \end{aligned}$$

and

$$|K(x,t,u(x,t)) - K(x,t,v(x,t))| \leq L(x,t)|u(x,t) - V(x,t)|.$$

for all $c, x, t \in I$ *and* $h, u, v \in C(I \times I)$. *Let* $y : I \times I \to \mathbb{R}$ *be a function such that*

$$|\mu(x,t)||p(x,t)y_{xx}(x,t) + q(x,t)y_x(x,t) - f(x,t,y(x,t))| \leq \varphi(x,t)$$

for all $x, t \in I$ *and the condition* (2.11) *holds. Then, There exists a unique solution* $y_0 : I \times I \to \mathbb{R}$ *of (2.10) such that*

$$|y(x,t) - y_0(x,t)| \leq \frac{\beta}{1-\beta}\varphi(x,t).$$

Proof. It follows from Theorem 2.2 that

$$y_0(x,t) = y(c,t) + \int_c^x K(\tau,t,y(\tau,t))\mathrm{d}\tau$$

with

$$K(x,t,y(x,t)) = -(\mu(x,t)p(x,t))^{-1}\Big[(\mu(x,t)q(x,t) - (\mu q)_x(x,t))y(x,t) + \left(h(c,t) - \int_c^x \mu(\tau,t)f(\tau,t,y(\tau,t))\mathrm{d}\tau\right)\Big]$$

and

$$h(c,t) = -[\mu(c,t)p(x,t)y_x(c,t) - (\mu p)_x(c,t)y(c,t) + \mu(c,t)q(c,t)y(c,t)]$$

has the required properties. This completes the proof. □

Remark 2.4. *In 2009, Jung* [7]*proved the Hyers-Ulam stability of linear partial differential equation of the first order of the following form:*

$$ay_x(x,t) + by_t(x,t) + g(x)y(x,t) + h(x) = 0$$

for all $a \geq 0$ *and* $b > 0$.

Now, we consider the generalization of this equation as follows:

$$ay_x(x,t) + by_t(x,t) = f(x,t,y(x,t)) \tag{2.12}$$

for all $a, b \in \mathbb{R}$ *with* $a \neq 0$ *and* $b \neq 0$. *Let* ζ *and* η *be defined by*

$$\zeta = x - \frac{a}{b}t, \quad \eta = \frac{1}{b}t. \tag{2.13}$$

If we define $\widetilde{y}(\zeta,\eta) = y(\zeta + a\eta, b\eta) = y(x,t)$, *then, by (2.13), we have*

$$y_x(x,t) = \widetilde{y}_\zeta(\zeta,\eta)\frac{\partial\zeta}{\partial x} + \widetilde{y}(\zeta,\eta)\frac{\partial\eta}{\partial x},$$
$$y_t(x,t) = \widetilde{y}_\zeta(\zeta,\eta)\frac{\partial\zeta}{\partial t} + \widetilde{y}_\eta(\zeta,\eta)\frac{\partial\eta}{\partial t} = -\frac{a}{b}\widetilde{y}_\zeta(\zeta,\eta) + \frac{1}{b}\widetilde{y}_\eta(\zeta,\eta).$$

Thus, we see that $ay_x(x,t) + by_t(x,t) = \widetilde{y}_\eta(\zeta,\eta)$, *and so we can rewrite the equation (2.12) as follows:*

$$\widetilde{y}_\eta(\zeta,\eta) = \widetilde{f}(\zeta,\eta,\widetilde{y}_\eta(\zeta,\eta)). \tag{2.14}$$

Now, we can use Theorem 2.1 for the generalized Hyers-Ulam stability of (2.14).

We consider the mixed type second order nonlinear partial differential equation:

$$\begin{aligned} &p(x,t)y_{xt}(x,t) + q(x,t)y_t(x,t) + p_t(x,t)y_x(x,t) - p_x(x,t)y_t(x,t) \\ &= f(x,t,y(x,t)). \end{aligned} \tag{2.15}$$

Now, we prove the Hyers-Ulam-Rassias stability of (2.15) under the condition:

$$p_{xt}(x,t) = q_t(x,t) \tag{2.16}$$

Theorem 2.5. *Let $c \in Ip$, $q : I \times I \to \mathbb{R}$ be continuous functions with $p(x, t) \neq 0$ for all x, $t \in I$, $\phi : I \times I \to (0, \infty)$ be a continuous function, $L : I \times I \to [1, \infty)$ be an integrable function, and $f : I \times I \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that there exists $0 < \beta < 1$ such that*

$$\int_c^x L(\tau,t)\varphi(\tau,t)\mathrm{d}\tau \leq \beta\varphi(x,t);$$

$$\begin{aligned} &h(x,c) = -[p(x,c)y_x(x,c) - p_x(x,c)y(x,c) + q(x,c)y(x,c)]; \\ &K(x,t,y(x,t)) \\ &= -(p(x,t))^{-1}\left[(p_x(x,t) - q(x,t))y(x,t) + h(x,c) - \int_c^t f(x,\tau,y(x,\tau)\mathrm{d}\tau\right] \end{aligned}$$

and

$$|K(x,t,u(x,t)) - K(x,t,v(x,t))| \leq L(x,t)|u(x,t) - v(x,t)|$$

for all c, x, $t \in I$ and h, y, u, $v \in C(I \times I)$. Let $y : I \times I \to \mathbb{R}$ be a function such that

$$\begin{aligned} &|p(x,t)y_{xt}(x,t) + q(x,t)y_t(x,t) + p_t(x,t)y_x(x,t) - p_x(x,t)y_t(x,t) - f(x,t,y(x,t))| \\ &\leq \varphi(x,t) \end{aligned} \tag{2.17}$$

for all x, t⎣ I and the condition (2.16) *holds. Then, there exists a unique solution $y_0 : I \times I \to \mathbb{R}$ of (2.15) such that*

$$|y(x,t) - y_0(x,t)| \leq \frac{\beta}{1-\beta}\varphi(x,t).$$

Proof. By (2.17) and (2.16), we see that

$$\begin{aligned} &|p(x,t)y_{xt}(x,t) + q(x,t)y_t(x,t) + p_t(x,t)y_x(x,t) - p_x(x,t)y_t(x,t) - f(x,t,y(x,t))| \\ &= |(p(x,t)y_x(x,t) - p_x(x,t)y(x,t) + q(x,t)y(x,t))_t \\ &\quad + [p_{xt}(x,t) - q_t(x,t)]y(x,t) - f(x,t,y(x,t))| \\ &= |(p(x,t)y_x(x,t) - p_x(x,t)y(x,t) + q(x,t)y(x,t))_t - f(x,t,y(x,t))| \end{aligned}$$

Thus, we have

$$\begin{aligned} &-\varphi(x,t) \\ &\leq (p(x,t)y_x(x,t) - p_x(x,t)y(x,t) + q(x,t)y(x,t))_t - f(x,t,y(x,t)) \\ &\leq \varphi(x,t). \end{aligned} \tag{2.18}$$

It follows from (2.18) that

$$\begin{aligned}
&\left| p(x,t)y_x(x,t) - p_x(x,t)y(x,t) + q(x,t)y(x,t) + h(c,t) - \int_c^t f(x,\tau,y(x,\tau))\mathrm{d}\tau \right| \\
&= |p(x,t)|^{-1} \Big| y_x(x,t) + (p(x,t))^{-1} \Big((q(x,t) - p_x(x,t))y(x,t) + h(c,t) \\
&\quad - \int_c^t f(x,\tau,y(x,\tau))\mathrm{d}\tau \Big) \Big| \\
&\le \int_c^t \varphi(x,\tau)\mathrm{d}\tau,
\end{aligned} \tag{2.19}$$

where

$$h(x,c) = -\left[p(x,c)y_x(x,c) - p_x(x,c)y(x,c) + q(x,c)y(x,c)\right].$$

From (2.19), we obtain

$$\begin{aligned}
&\left| y_x(x,t) + (p(x,t))^{-1} \left((q(x,t) - p_x(x,t))y(x,t) + h(c,t) - \int_c^t f(x,\tau,y(x,\tau))\mathrm{d}\tau \right) \right| \\
&\le |p(x,t)|^{-1} \int_c^t \varphi(x,\tau)\mathrm{d}\tau.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.2. This completes the proof. □

Remark 2.6. *We can define the integrating factor for the equation (2.15) and prove a corollary similar to Corollary 2.3 for Theorem 2.6.*

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Authors' contributions
AB carried out the molecular genetic studies, participated in the sequence alignment and drafted the manuscript. JY carried out the immunoassays. MT participated in the sequence alignment. ES participated in the design of the study and performed the statistical analysis. FG conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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